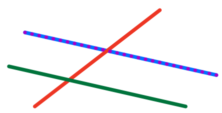
1.1 : Plane Curves

Dfn: Line $L \subset \mathbb{C}^2$ given by solutions of the equation
$ax + by + c = 0$, $(x,y) \in \mathbb{C}^2$ $a,b,c \in \mathbb{C}$ $(a,b) \neq (0,0)$.

Prop: $P_1, P_2 \in \mathbb{C}^2$ distinct. Then $\exists!$ Line passing through them.
equation of line is given by: $\det\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} = 0$

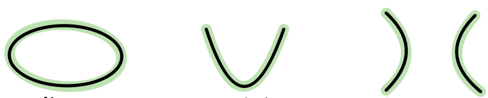
Prop : $L_1 \cap L_2$ in $\mathbb{C}^2$. Then one of these is true:
- $L_1 = L_2$
- $L_1 \cap L_2 = \{P\}$, $P \in \mathbb{C}^2$,
- $L_1 \cap L_2 = \emptyset$ (parallel)



Dfn: Conic $C \subset \mathbb{C}^2$ is a plane curve given by
$q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$
$(a,b,c) \neq (0,0,0)$.
Irreducible: if polynomial does not factor.

Rem: Reducible conic:
$ax^2 + bxy + cy^2 + dx + ey + f = (\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2)$.
= union of two lines.

in $\mathbb{R}^2$: (nondeg)



ellipse parabola hyperbola

and other degenerate cases.

Prop: L a line and C a conic. Then either $C = L \cup L_2$ for some line L_2 (C is reducible) or $|L \cap C| \leq 2$.

Dfn: Plane curve $C \subset \mathbb{C}^2$ given by equation
$p(x,y) = 0$, $p(x,y)$ is non-constant poly.

1.2 Projective Curves

Dfn: Complex projective line $\mathbb{P}^1 = (\mathbb{C}^2 \setminus (0,0))/\sim$,
where $x \sim \lambda x$ $\forall\, 0 \neq \lambda \in \mathbb{C}$.

homogeneous coordinates: $[x:y] = [\lambda x : \lambda y]$.

Rem: $[x:y] \in \mathbb{P}^1$ gives line $ax + by = 0 \subset \mathbb{C}^2$ through origin.

$\mathbb{P}^1 = \mathbb{C}$ "with an extra point at infinity".

Dfn: Complex projective plane $\mathbb{P}^2 = (\mathbb{C}^3 \setminus (0,0,0))/\sim$
where $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ $\forall\, 0 \neq \lambda \in \mathbb{C}$.
$[x:y:z] = [\lambda x : \lambda y : \lambda z]$.

Cor: $\mathbb{P}^2 = \mathbb{C}^2$ with a disjoint $\mathbb{P}^1$ (line at infinity)

Dfn: Projective curve $C \subset \mathbb{P}^2$ given by the equation
$p(x,y,z) = 0$,
where p is a nonzero homogeneous polynomial.

Dfn: Line $L \subset \mathbb{P}^2$ given by solutions of the equation
$ax + by + cz = 0$ $[x:y:z] \in \mathbb{P}^2$,
where $(a,b,c) \neq (0,0,0)$.

Prop: $P, Q \in \mathbb{P}^2$, $P \neq Q$. Then $\exists!$ line passing through P and Q
$P = [x_1 : y_1 : z_1]$, $Q = [x_2 : y_2 : z_2]$, equation is given by:
$$\det\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 0$$

Prop: $L_1, L_2 \subset \mathbb{P}^2$. Then either $L_1 = L_2$ or $|L_1 \cap L_2| = 1$.

Dfn: Conic $C \subset \mathbb{P}^2$ is given by solutions of
$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$
where $(a,b,c,d,e,f) \neq (0,0,0,0,0,0)$

Prop: $L, C \subset \mathbb{P}^2$. Then either $C = L \cup L_2$ for some L_2, or $|L \cap C| = 1$ or 2.

1.3 Projective Transformations

Affine transformation: $\mathbb{C}^2 \to \mathbb{C}^2$: $T(x) = Ax + B$, where
$A \in GL_2(\mathbb{C})$ and B a translation vector.
↳ Euclidean := A orthogonal

Projective Transformation: $\mathbb{P}^2 \to \mathbb{P}^2$: $T(X) = MX$, $M \in GL_3(\mathbb{C})$.
$PGL_3(\mathbb{C}) = GL_3(\mathbb{C})/\mathbb{C}^*$

identity map $= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$

Thm: $P_1, P_2, P_3, P_4 \in \mathbb{P}^2$, such that no three are collinear. Then $\exists$ projective transformation $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ s.t.
$\phi(P_1) = [1:0:0]$ $\phi(P_3) = [0:0:1]$
$\phi(P_2) = [0:1:0]$ $\phi(P_4) = [1:1:1]$.

$P_i = [x_i : y_i : z_i]$. Then $\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$ is a projective trans.
giving $\varphi([1:0:0]) = P_1$, $\varphi([0:1:0]) = P_2$, $\varphi([0:0:1]) = P_3$.

Thm: $P_1, \ldots, P_5 \in \mathbb{P}^2$ distinct, and no 3 collinear. Then $\exists!$ conic passing through them.

1.4 Classification of Conics

Thm: $C \subset \mathbb{P}^2$ a conic. Then $\exists$ a projective trans. $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\phi(C)$ is one of the following:

① $x^2 + y^2 + z^2 = 0$ (irreducible) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

② $x^2 + y^2 = 0$ ($L_1 \cup L_2$) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

③ $x^2 = 0$ (double line) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Slides show how to find such a transformation.

eqn of conic $C \subset \mathbb{P}^2$, $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ can be written in symmetric matrix form:

$$B = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

Prop: conic $C \subset \mathbb{P}^2$ is irreducible iff $\det(B) \neq 0$.

Thm (Intersection of Conics)

Suppose $C, C' \subset \mathbb{P}^2$ are two unequal irreducible conics. Then $1 \leq |C \cap C'| \leq 4$.

2.1 Smoothness, Tangents and Transversality

Dfn: $C \subset \mathbb{P}^2$ an irreducible curve, $P \in C$ is *smooth* if the gradient

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)\bigg|_P \neq (0,0,0)$$

Smooth curve: P smooth $\forall P \in C$.

$\mathrm{Sing}(C)$ = set of singular points of C.

Dfn: $f(x,y,z)$ irred. curve $\subset \mathbb{P}^2$, $P = [\alpha:\beta:\gamma]$ a smooth point.

Then the *tangent line at P*:

$$\frac{\partial f}{\partial x}(\alpha,\beta,\gamma)x + \frac{\partial f}{\partial y}(\alpha,\beta,\gamma)y + \frac{\partial f}{\partial z}(\alpha,\beta,\gamma)z = 0$$

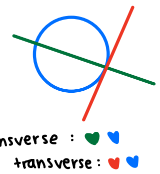
Lem: a line is smooth

Lem: an irreducible conic is smooth.

Dfn: $C_1, C_2 \subset \mathbb{P}^2$ curves intersecting at P. This intersection is called *transverse* if

① P is smooth in both C_1 and C_2

② Tangent lines of P in C_1 and C_2 differ.



Prop: $L \subset \mathbb{P}^2$ a line and C an irreducible conic.

Then either L is tangent to C ($|L \cap C| = 1$), or L intersects C transversely at 2 points.

Prop: $C_1, C_2 \subset \mathbb{P}^2$ distinct, irred. conics. Then they intersect transversely at all points $\Leftrightarrow$ there are 4 intersection points.

2.2 Bezout's Theorem

Thm: $f(x,y)$ nonzero homogeneous poly of deg d. Then $f(x,y) = 0$ in $\mathbb{P}^1$ consists of d points counted with multiplicity.

Bezout's Theorem:

$f(x,y,z)$, $g(x,y,z)$ hom. poly without common factors.

Then the solutions of the system $\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$ in $\mathbb{P}^2$ are given by $\deg(f)\deg(g)$ points counted with multiplicity.

Bezout's Theorem: C_1, C_2 distinct irred. curves with degrees d_1 and d_2. Then there are $d_1 d_2$ intersection points counted with multiplicity.

Rem: common factor $\Rightarrow$ infinitely many solutions (in $\mathbb{P}^2$)

Dfn (Intersection multiplicity at P). Assume $P = [0:0:1]$:

$R = \mathbb{C}[\bar{x},\bar{y}]_{(0,0)}$:= commutative ring of quotients of polynomials $\frac{a(\bar{x},\bar{y})}{b(\bar{x},\bar{y})}$ where $b(0,0) \neq 0$.

$I \subset R$:= ideal generated by $f(\bar{x},\bar{y},1)$ and $g(\bar{x},\bar{y},1)$.

Intersection multiplicity $= (f,g)_P := \dim_{\mathbb{C}}(R/I)$

or use $R = \mathbb{C}[[\bar{x},\bar{y}]]$:= ring of power series

Properties of intersection multiplicities:

Say $f(P) = g(P) = h(P) = 0$. Then

① $(f,g)_P \geq 1$

② $\Rightarrow (f,gh)_P = (f,g)_P + (f,h)_P$

③ If $h(P) \neq 0$, then $(f,gh)_P = (f,g)_P$

④ Intersection at P transverse $\Leftrightarrow (f,g)_P = 1$.

Rem: transverse $\forall$ intersection points $\Rightarrow |C_1 \cap C_2| = d_1 d_2$

Write an irreducible curve $C \subset \mathbb{P}^2$ of degree d as

$$z^d h_0(x,y) + z^{d-1} h_1(x,y) + \dots + h_d(x,y) = 0$$

h_i = hom. poly of deg = i.

Dfn: $P = [0:0:1]$. Multiplicity $\mathrm{mult}_P(C)$ is the smallest n such that $h_n(x,y)$ is nonzero.

① $P \in C \Leftrightarrow \mathrm{mult}_P(C) \geq 1$

② P is a singular point $\Leftrightarrow \mathrm{mult}_P(C) \geq 2$. !

Prop: C_1, C_2 distinct, irred. curves in $\mathbb{P}^2$ and $P \in C_1 \cap C_2$. Then

$$(C_1 \cdot C_2)_P \geq \mathrm{mult}_P(C_1) \cdot \mathrm{mult}_P(C_2)$$

2.3. Applications of Bezout's Theorem

Prop: $f(x,y,z)$ hom. poly. If the system

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

has no solutions in $\mathbb{P}^2$, then f is irreducible.

Prop: C irred. curve of deg $d \geq 2$ and $P, Q \in C$ distinct.

Then $\mathrm{mult}_P(C) + \mathrm{mult}_Q(C) \leq d$

Rem: For a line L and curve C with deg $d \geq 2$,

Bezout $\Rightarrow$ $d = \sum_{R \in L \cap C} (L \cdot C)_R$, and $(L \cdot C)_P \geq \mathrm{mult}_P(C)$.

Cor: $P \in C$ on irred curve, deg $d \geq 2$. Then $\mathrm{mult}_P(C) < d$.

Cor: An irreducible cubic has at most 1 singular point.

2.4 Points on Curves

Prop: $P_1, P_2 \in \mathbb{P}^2$ distinct. $\exists!$ line passing through them.

Prop: $P_1, \dots, P_5 \in \mathbb{P}^2$ distinct, no four contained in a line. Then $\exists!$ Conic passing through them.

Prop: let $C \subset \mathbb{P}^2$ be irred. curve, deg ≥ 4. Then C has at most 3 singular points.

$S_d :=$ space of homogeneous degree d polynomials
$\dim(S_d) = \frac{(d+1)(d+2)}{2}$

Dfn: Σ a finite set of points. Then
$$S_d(\Sigma) := \{ f \in S_d \mid f(P) = 0 \ \forall P \in \Sigma \}$$

"Σ imposes independent conditions on $S_d \Leftrightarrow$
$\dim S_d(\Sigma) = \dim S_d - |\Sigma|$.

note: $\dim S_d(\Sigma) \geq \dim S_d - |\Sigma|$.

$\dim S_1 = 3$

- P imposes i.c. on S_1
- P, Q impose i.c. on $S_1 \Leftrightarrow P \neq Q$
- P, Q, R impose i.c. $\Leftrightarrow P, Q, R$ are not collinear.
- ≥ 4 points do not impose i.c.s.

$\dim S_2 = 6$

- $1, \dots, 3$ points impose ics $\Leftrightarrow$ distinct
- 4 points impose ics $\Leftrightarrow$ distinct and not collinear
- 6 points impose ics $\Leftrightarrow$ do not lie on a conic
- ≥ 7 fail to impose ics.

Thm: 5 points impose i.c.s on S_2 (i.e. $\exists!$ conic passing through them) $\Leftrightarrow$ no 4 are collinear

See slides for finding i.c. Criteria.

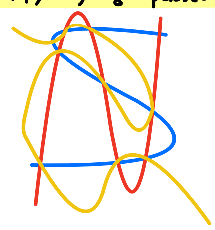
Prop: $\Sigma \subset \mathbb{P}^2$.

① Suppose $a > d$ points lie on a line L given by $f(x,y,z) = 0$. Then $S_d(\Sigma) = f \cdot S_{d-1}(\Sigma \setminus L)$

② Suppose $a > 2d$ points lie on an irreducible conic C given by $f(x,y,z) = 0$. Then $S_d(\Sigma) = f \cdot S_{d-2}(\Sigma \setminus C)$.

Thm: $P_1, \dots, P_8 \in \mathbb{P}^2$ distinct, and suppose at most 3 lie on a line and at most 6 lie on an irred. conic. Then $P_1, \dots, P_8$ impose ics on S_3.

Thm: (Chasles) C_1, C_2 cubics intersecting at 9 distinct points, $P_1, \dots, P_9$. Then any cubic passing through $P_1, \dots, P_8$ passes through P_9.



Thm: (Pascal) Let C be an irreducible Conic and $P_1, P_2, P_3, Q_1, Q_2, Q_3$ distinct points on C. Then $R_1 = P_1Q_1 \cap P_3Q_3$, $R_2 = P_2Q_1 \cap P_3Q_2$, and $R_3 = P_2Q_3 \cap P_1Q_2$ are collinear.

3.1 Inflection points

Rem: $C \subset \mathbb{P}^2$ curve, L line intersecting C at smooth point. $(L \cdot C)_p \geq 2 \Leftrightarrow L$ is tangent at P.

Dfn: $P \in C$ is called *inflection point* if it is smooth and $(L \cdot C)_P \geq 3$, L = tangent at P.

Dfn: $f(x,y,z)$ hom. poly. *Hessian:*

$$\mathrm{Hess}(f) = \det \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix} \qquad f_{ij} = \frac{\partial^2 f}{\partial i \, \partial j}$$

Thm (Hess. Criterion): Let $P \in \mathbb{P}^2$ satisfy $f(P) = \mathrm{Hess}(f)(P) = 0$. If $p \in C$, $C = f(x,y,z)$ smooth, then P is an inflection point. $\Rightarrow (f \cdot \mathrm{Hess}(f))_p = 1 \Leftrightarrow (L \cdot f)_p = 3$, L = tangent at P.

Prop: Let f have no linear factors. Then $f=0$ has finitely many inflection points. $\Rightarrow \mathrm{Hess}(f)$ and f have no common factors.

Prop: $C \subset \mathbb{P}^2$ smooth curve, $\deg \geq 3$. Then C has at least one inflection point.

Prop: $C \subset \mathbb{P}^2$ smooth cubic. Then C has 9 distinct inflection points.

3.2 : Classification of Cubics

Rem: any line $L \subset \mathbb{P}^2$ is projectively equivalent to $z=0$

Thm (Weierstrass form): Let $C \subset \mathbb{P}^2$ be a smooth cubic. Then $\exists$ projective transformation which takes it to

$$y^2 z = x^3 + axz^2 + bz^3$$

Thm: A weierstrass cubic $y^2 z = x^3 + axz^2 + bz^3$ is smooth iff the discriminant

$$\Delta = -16(4a^3 + 27b^2) \neq 0$$

Rem: $\Delta = 0 \Leftrightarrow x^3 + axz^2 + bz^3$ has a repeated root.

Thm (Legendre form): $C \subset \mathbb{P}^2$ smooth cubic. Then $\exists$ projective transformation which takes it to the form

$$y^2 z = x(x-z)(x-\lambda z)$$

for $\lambda \neq 0, 1$.

Dfn: The *j-invariant* of $y^2 z = x^3 + axz^2 + bz^3$ is

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

Thm: two smooth Weierstrass cubics are projectively equivalent iff they have the same j invariant.

Thm: $C \subset \mathbb{P}^2$ singular irreducible cubic. Then $\exists$ projective transformation taking C to one of the following forms:

① (nodal) $zy^2 = x^2(x+z)$

② (cuspidal) $zy^2 = x^3$

Thm: *reducible cubic always contains a line.*

Projectively equivalent to

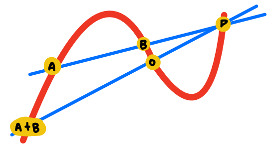
$x(zy + x^2) = 0$	$xy(x+y) = 0$
$x(zx + y^2) = 0$	$x^2 y = 0$
$xyz = 0$	$x^3 = 0$

3.3. Group Law on Elliptic curves

Dfn: Elliptic curve :: smooth cubic $E \subset \mathbb{P}^2$ with a chosen point $O \in E$.

$E \subset \mathbb{P}^2$ elliptic, $A, B \in E$. Define $A+B \in E$ as follows:

- $L_1 := AB$. If $A = B$, L_1 = tangent line at A.
- $L_1 \cap E = \{A, B, P\}$ counted w/ multiplicity. Degen. cases:
  - $A = B \neq P$. L_1 tangent to A, intersects P transversely
  - $A \neq B = P$. L_1 tangent to P, intersects A transversely
  - $A = B = P$. L_1 tangent to A, A an inflection point.
- $L_2 := OP$. Third intersection point is $A+B$.



$\Rightarrow E$ forms an abelian group.

Prop: $A, B, C \in E$, and $O \in E$ an inflection point. Then A, B, C lie on a line $\Leftrightarrow A+B+C=0$

Prop: $O \in E$ an inflection point and $P \in E$. The inverse $-P \in E$ is the third point on line OP.

Prop: $O \in E$ an inflection point, $A \in E$ s.t $3A = 0$. Then A is an inflection point.

Denote third point on Line OP by $\bar{P}$

If O an inflection point, then $-A = \bar{A}$

4.1 Elliptic curves over other Fields

Dfn: K a field. $C \subset \mathbb{P}^2$ given by $f(x,y,z) = 0$. Then $C(K) := \{ [x:y:z] \in K : f(x,y,z)\} = 0$.

Rem: Cubic smooth over $\mathbb{F}_p$ iff $p \nmid \Delta$

Prop: $E(K)$ an abelian group.

Thm (Mordell): $E(\mathbb{Q}) \cong \mathbb{Z}^{\text{rank}} \times \mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_n\mathbb{Z}$

Thm (Faltings): $C \subset \mathbb{P}^2_{\mathbb{Q}}$ smooth curve deg $d \geq 4$. Then $C(\mathbb{Q})$ is finite.

Dfn: G abelian, $a, b \in G$. *Discrete logarithm:* $\log_b a \in \mathbb{Z}$ s.t $a = b^{\log_b a}$.

Prop: $E : y^2 z = x^3 + axz^2 + bz^3$. Then $E(\mathbb{F}_p) \leq 2p+1$.

4.2. Rational Curves

Dfn: $C \subset \mathbb{P}^2$ is *rational* if $\exists$ non-constant map $\mathbb{P}^1 \to \mathbb{P}^2$ $[a:b] \mapsto [p(a,b) : q(a,b) : r(a,b)]$ for some hom. poly of the same deg ≥ 1, whose image is contained in C.

Prop: C rational $\Leftrightarrow \exists\ P \in C(\mathbb{C}(t))$ with nonconstant coords.

Prop: C irred. + rational $\Rightarrow \mathbb{P}^1 \to C$ surjective

Prop: C irred conic $\Rightarrow \exists$ iso $\mathbb{P}^1 \to C \Rightarrow C$ rational

Prop: C irred. singular cubic $\Rightarrow C$ rational

- cuspidal: $[a:b] \mapsto [a^2 b : a^3 : b^3]$
  nodal: $[a:b] \mapsto [a^2 b - b^3 : a^3 - ab^2 : b^3]$

Prop: Legendre cubic is not rational
$\Rightarrow$ smooth cubics are not rational.

4.3 : Topology of Curves

Thm: every real 2D connected, compact, oriented manifold is homeomorphic to a compact, oriented surface of genus $g \geq 0$

Thm (Genus-degree formula)
$C \subset \mathbb{P}^2$ smooth curve of deg d. Then C is homeomorphic to a compact, oriented surface of genus $g = \frac{(d-1)(d-2)}{2}$

relative to quotient topology

5.1 Noetherian Rings

Dfn: Ring *noetherian* if all its ideals are finitely generated

Prop: R noetherian, $I \subset R$ ideal $\Rightarrow$ R/I Noetherian.

Thm: R Noetherian
$\Leftrightarrow$ Every ascending chain of ideals stabilizes
$\Leftrightarrow$ Every non $\emptyset$ set of ideals in R has a max element

Thm: R Noetherian $\Rightarrow$ $R[x]$ Noetherian.
... $\mathbb{C}[x_1,\dots,x_n]$ Noetherian.

5.2: Algebraic Sets

Dfn: $\Sigma \subset \mathbb{C}^n$. *Vanishing ideal* $I(\Sigma) \subset \mathbb{C}[x_1,\dots,x_n]$ is the ideal of poly f s.t $f(P) = 0 \ \forall P \in \Sigma$.

Rem: $\Sigma_1 \subset \Sigma_2 \Rightarrow I(\Sigma_2) \subset I(\Sigma_1)$
$I(\Sigma) = \mathbb{C}[x_1,\dots,x_n] \Leftrightarrow \Sigma = \emptyset$.

Dfn: $I \subset \mathbb{C}[x_1,\dots,x_n]$ ideal. *Vanishing set* $V(I) \subset \mathbb{C}^n$ $= \{P \in \mathbb{C}^n : f(P) = 0 \ \forall f \in I\}$. Called "*Algebraic set*"

Rem: $I_1 \subset I_2 \Rightarrow V(I_2) \subset V(I_1)$

Lem: $V(I(\Sigma)) = \Sigma$ **Lem:** $I \subset I(V(I))$

Dfn: $I \subset \mathbb{C}[x_1,\dots,x_n]$ an ideal. Radical $\sqrt{I} \subset \mathbb{C}[x_1,\dots,x_n]$ $\{f \in \mathbb{C}[x_1,\dots,x_n] : f^n \in I\}$. I "*radical*" if $\sqrt{I} = I$.

Rem: prime ideal: $\forall\, ab \in I$, then $a \in I$ or $b \in I$.
is radical: $f^m \in I$, then $f \in I$ or $f^{m-1} \in I$

Rem: $V(\sqrt{I}) = V(I)$

Thm (Nullstellensatz): $I \subset \mathbb{C}[x_1,\dots,x_n]$:
$$\sqrt{I} = I(V(I))$$

Cor (W.N 1): $m \subset \mathbb{C}[x_1,\dots,x_n]$ maximal. Then $m = (x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1,\dots,a_n) \in \mathbb{C}^n$

Cor (W.N. 2) $V(I) = \emptyset \Rightarrow I = \mathbb{C}[x_1,\dots,x_m]$

Cor: $\exists$ 1:1 correspondence

$$V: \{\text{radical ideals}\} \rightleftarrows \{\text{algebraic subsets}\} : I$$

Dfn: alg. subset is *irreducible* if it is not a union of two distinct alg. subsets.

Cor $\exists$ 1:1 correspondence

$$V: \{\text{prime ideals}\} \rightleftarrows \{\text{irred. algebraic subsets}\} : I$$

6.1 Polynomial functions

Dfn: *polynomial function* $f: \mathbb{C}^n \to \mathbb{C}$; $(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$ for $f \in \mathbb{C}[x_1, \dots, x_n]$.

Dfn: let $X \subset \mathbb{C}^n$ be an algebraic subset. The *coordinate ring* is $\mathbb{C}[X] := \mathbb{C}[x_1, \dots, x_n] / I(X)$

Rem: $\mathbb{C}[\mathbb{C}^n] = \mathbb{C}[x_1, \dots, x_n]$

Dfn: $X \subset \mathbb{C}^m$, $Y \subset \mathbb{C}^n$. Then X and Y are *isomorphic* if $\mathbb{C}[X] \cong \mathbb{C}[Y]$ as rings.

Lem: line $L \subset \mathbb{C}^2$. Then $L \cong \mathbb{C}$ as an algebraic set.

Useful: R/I is an integral domain iff I is a prime ideal.

Dfn: a commutative ring is *reduced* if $f^N = 0$ for $f \in R$ implies that $f = 0$.

Rem: integral domain is reduced.

Lem: $R = \mathbb{C}[x_1, \dots, x_n]/I$ is reduced $\Leftrightarrow$ I is radical.

Thm: $\exists$ 1:1 correspondence

$\{$affine algebraic sets$\}/\{$isomorphism$\}$

$\updownarrow$

$\{$reduced finitely generated $\mathbb{C}$-algebras$\}/\{$isomorphism$\}$

given by sending $X \mapsto \mathbb{C}[X]$.

6.2. Affine Varieties

Dfn: An *affine variety* is an irreducible affine algebraic set.

Thm: $\exists$ 1:1 correspondence

$\{$affine varieties$\}/\{$isomorphism$\}$

$\updownarrow$

$\{$fin. gen., integral domain $\mathbb{C}$-algebras$\}/\{$isomorphism$\}$

Prop: A prime ideal $I \subset \mathbb{C}[x, y]$ is either:

- $I = 0$
- $I = (f)$ for an irreducible polynomial $f \in \mathbb{C}[x, y]$
- $I = \langle x - a, y - b \rangle$ for $a, b \in \mathbb{C}$.

Lem: Cuspidal cubic $C: y^2 = x^3$ is irreducible

Prop: $C \ncong \mathbb{C}$ as an affine variety

6.3. Polynomial Maps and Normal Varieties

Dfn: $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ algebraic sets. A *polynomial map* $f: X \to Y$ is a map given by $f: P \mapsto (f_1(P), \dots, f_m(P))$ for some $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$.

Lem: $f: X \to Y$ poly. map induces map of $\mathbb{C}$-algebras by $f^*: \mathbb{C}[Y] \to \mathbb{C}[X]$; $g \mapsto g \circ f$, that is contravariant: $f^* \circ g^* = (g \circ f)^*$.

Lem: $F: \mathbb{C}[Y] \to \mathbb{C}[X]$ homo. Then $F = f^*$ for a $!$ poly $f: X \to Y$.

Dfn: poly map $f: X \to Y$ is an *isomorphism* if $\exists$ poly map $g: Y \to X$ s.t. $g \circ f = id_X$ and $f \circ g = id_Y$.

Dfn: R an I.D. and K its field of fractions

- $\alpha \in K$ is *integral* over R if $\exists$ $a_0, \dots, a_{d-1} \in R$ such that $\alpha^d + a_{d-1}\alpha^{d-1} + \dots + a_0 = 0$
- The *integral closure* $\bar{R} \subset K$ is the set of elements integral over R
- R is *integrally closed* if $\bar{R} = R$.

Dfn: aff. var. X is *normal* if $\mathbb{C}[X]$ is integrally closed. $\mathbb{C}(X)$ = field of rational fractions on X.

Prop: UFD is integrally closed.

Ex. $\mathbb{C}^n$ is normal.

Thm (Zariski): a curve $C \subset \mathbb{C}^2$ is smooth $\Leftrightarrow$ C is normal

6.4: Zariski Topology on $\mathbb{C}^n$

Dfn: a *Zariski closed subset* $Z \subset \mathbb{C}^2$ is an algebraic subset.

Prop: $\hookrightarrow$ forms topology on $\mathbb{C}^2$.

Rem: can intersect Zariski closed subsets with alg. set $X \subset \mathbb{C}^n$ to define topology on X.

Prop: poly function $f: X \to Y$ continuous in Zariski top.

6.5: Automorphisms

X = aff. alg set. $Aut(X) := \{$isomorphisms $X \to X\}$. = group.

Prop: $Aut(\mathbb{C})$ is isomorphic to the group of affine transformations $x \mapsto ax + b$, $a \neq 0$.

Thm (Jung): $Aut(\mathbb{C}^2)$ is gen. by $(x, y) \mapsto (x, y + f(x))$ $f \in \mathbb{C}[x]$, and $(x, y) \mapsto (ax + by + \alpha, cx + dy + \beta)$, $ad - bc \neq 0$.

7.1 Rational Functions

Dfn: elements of $\mathbb{C}(X)$ are called *rational functions*

Dfn: $\phi \in \mathbb{C}(X)$ is *regular* at $P \in X$ if ϕ can be written as $\frac{f}{g}$ with $g(P) \neq 0$. $\mathrm{dom}(\phi) := \{ P \in X$ where ϕ is regular$\}$

Dfn: $K \subset L$ a field

- $S \subset L$ is *algebraically independent* over K if elements in S do NOT satisfy a single nontrivial polynomial equation with coeffs. in K.
- The *transcendence degree* of L/K is the largest cardinality of an alg. indep. subset of L over K.
- X aff. var. The *dimension* of X is tr. deg. of $\mathbb{C}(X):\mathbb{C}$

7.2 Rational maps

Dfn: *rational map* $f: X \dashrightarrow \mathbb{C}^m$ is a collection of rational functions $f_1, \dots, f_m \in \mathbb{C}(X)$. $\mathrm{dom}(f) = \cap_{i=1}^m \mathrm{dom}(f_i)$

Dfn: *rational map* $f: X \dashrightarrow Y$ = $f: X \to \mathbb{C}^m$ s.t. $f(\mathrm{dom}(f)) \subset Y$

Thm: $\phi: X \dashrightarrow Y$ rational map of aff. var.

① $\mathrm{dom}(\phi) \subset X$ is open and dense in Zariski topology

② $\mathrm{dom}(\phi) = X \iff \phi$ is a polynomial map (defined everywhere)

Dfn: $\phi: X \dashrightarrow Y$ is *dominant* if $\phi(\mathrm{dom}\,\phi)$ is dense in Y

Prop: $\phi: X \dashrightarrow Y$ dominant, $\psi: Y \dashrightarrow Z$ arbitrary, then $\psi \circ \phi: X \dashrightarrow Z$ is well defined.

X, Y aff. var, $\phi: X \dashrightarrow Y \Rightarrow \phi^*: \mathbb{C}[Y] \to \mathbb{C}(X)$, $\phi^*: f \mapsto f \circ \phi$.

ϕ dominant $\Rightarrow \phi^*$ injective.

Cor: $\phi: X \dashrightarrow Y$ dominant, induces homo $\phi^*: \mathbb{C}(Y) \to \mathbb{C}(X)$.

Lem: $\Phi: \mathbb{C}(Y) \to \mathbb{C}(X)$ homo of $\mathbb{C}$-alg. Then $\exists!$ dominant map $\phi: X \dashrightarrow Y$ s.t. $\Phi = \phi^*$.

Dfn: dominant $\phi: X \dashrightarrow Y$ is *birational* if $\exists$ dominant $\psi: Y \dashrightarrow X$ s.t. $\psi \circ \phi = id_X$, $\phi \circ \psi = id_Y$.

Cor: X, Y birational $\iff \mathbb{C}(X) \cong \mathbb{C}(Y)$ as $\mathbb{C}$-algebras.

7.3 Projective Nullstellensatz

Dfn: an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is *homogeneous* if for every $f \in I$, its homogeneous components also lie in I.

Lem: $I \subset \mathbb{C}[x_0, \dots, x_n]$ homo $\iff I$ gen. by homo.

Dfn: $I \subset \mathbb{C}[x_0, \dots, x_n]$ homo. The *vanishing set* is $V(I) := \{ P \in \mathbb{P}^n : h(P) = 0 \ \forall h \in I, h \text{ homo}\}$

$V(I)$ is called *algebraic*.

Rem: $\mathbb{C}[x_0, \dots, x_n]$ Noetherian $\Rightarrow I$ fin. gen.

$V(I)$ = zero locus of some finite set of polynomials.

Dfn: $X \subset \mathbb{P}^n$ a subset. The *ideal of vanishing* is $I(X) := \{ h \in \mathbb{C}[x_0, \dots, x_n]$ homo : $h(P) = 0 \ \forall P \in X\}$

Thm: (Projective Nullstellensatz) I homo, then

① $V(I) = \emptyset \iff \langle x_0, \dots, x_n \rangle \subset \sqrt{I}$

② $V(I) \neq \emptyset \Rightarrow \sqrt{I} = I(V(I))$

Cor: $\exists$ 1:1 correspondence

$$\{\text{hom. radical ideals of } \mathbb{C}[x_0, \dots, x_n] \text{ not containing } \langle x_0, \dots, x_n \rangle\}$$
$$\updownarrow$$
$$\{\text{algebraic subsets of } \mathbb{P}^n\}$$

Dfn: Alg. sub. $X \subset \mathbb{P}^n$ is *irreducible* if $X \neq X_1 \cup X_2$, X_1, X_2 alg subs. Called a *projective variety*.

Cor: $\exists$ 1:1 correspondence

$$\{\text{hom. prime ideals of } \mathbb{C}[x_0, \dots, x_n] \text{ not containing } \langle x_0, \dots, x_n \rangle\}$$
$$\updownarrow$$
$$\{\text{irred. algebraic subsets of } \mathbb{P}^n\}$$

Dfn: $X \subset \mathbb{P}^n$ *closed* if algebraic. $\to$ defines topology.

$\mathbb{P}^n$ covered by open sets $U_i \subset \mathbb{P}^n$ defined by $x_i \neq 0$. Gives iso $U_i \cong \mathbb{C}^n$ $[x_0 : \dots : x_n] \mapsto (x_0/x_i, \dots, \hat{1}, \dots, x_n/x_i)$.

7.4. Birational Maps

Dfn: **rational function** $\phi : X \dashrightarrow \mathbb{C}$:= $\phi = \frac{f}{g}$, $f, g \in \mathbb{C}[x_0, \dots, x_n]$ hom., $\deg(f) = \deg(g)$, $g \notin I(X)$

- ϕ **regular** at $P \in X$ if $g(P) \neq 0$.
- $\text{dom}(\phi) = \{ P \in X : P \text{ regular} \}$

Lem: U_i standard open $\subset \mathbb{P}^n$.
$X \cap U_i \neq \emptyset \Rightarrow \mathbb{C}(X) \cong \mathbb{C}(X \cap U_i)$

Cor: $X \cap U_i \neq \emptyset \neq X \cap U_j$, then $X \cap U_i$ and $X \cap U_j$ are birational as affine varieties.

Dfn: $\phi : X \dashrightarrow Y$ a **morphism** (poly map) if $\text{dom}(\phi) = X$.

Dfn: $\phi : X \dashrightarrow Y$ **dominant** if $\phi(\text{dom}\,\phi)$ is dense in Y

Dfn: $\phi : X \dashrightarrow Y$ **birational** if $\exists$ dom. $\psi : Y \dashrightarrow X$ $\begin{cases} \psi \circ \phi = id_X \\ \phi \circ \psi = id_Y \end{cases}$

Cor: $\mathbb{C}(X) \cong \mathbb{C}(Y) \Leftrightarrow X, Y$ birational.

Dfn: X proj. var is **rational** if X birat. to $\mathbb{P}^n$

Ex. smooth conic is rational

Ex. Legendre cubic rational $\Leftrightarrow \lambda = 0, 1$.

8.1 Surfaces

Dfn: *surface* $S \subset \mathbb{P}^3$ of deg d := zero set of hom poly $f(x,y,z,w) = 0$, $\deg f = d$.

Dfn: a *projective transformation* is an iso $\mathbb{P}^3 \to \mathbb{P}^3$ given by acting on $[x:y:z:w]$ by an invertible 4x4 matrix
$PGL_4(\mathbb{C}) := GL_4(\mathbb{C})/\mathbb{C}^*$

Dfn: $S \subset \mathbb{P}^3$ is *smooth* if $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right)\Big|_p \neq 0 \; \forall p \in S$.

Dfn: $P \in S$ smooth. The *tangent plane* at P is:

$$\frac{\partial f}{\partial x}(P)\,x + \frac{\partial f}{\partial y}(P)\,y + \frac{\partial f}{\partial z}(P)\,z + \frac{\partial f}{\partial w}(P)\,w = 0$$

Dfn: line $\subset \mathbb{P}^3$ is an embedding $\mathbb{P}^1 \subset \mathbb{P}^3$ given by:
$[\lambda:\mu] \mapsto [x_1\lambda + x_2\mu : y_1\lambda + y_2\mu : z_1\lambda + z_2\mu : w_1\lambda + w_2\mu]$
$[x_1:y_1:z_1:w_1] \neq [x_2:y_2:z_2:w_2]$

Prop: $P \in S$ smooth, $p \in \ell \subset S$. Then $\ell \subset$ tangent plane at P.

8.2. Quadric Surfaces

Dfn: *quadric surface* $\subset \mathbb{P}^3$ is given by
$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gxw + Hyw + Izw + Jw^2 = 0$

equivalently:

$$(x,y,z,w)\underbrace{\begin{pmatrix} A & B/2 & D/2 & G/2 \\ B/2 & C & E/2 & H/2 \\ D/2 & E/2 & F & I/2 \\ G/2 & H/2 & I/2 & J \end{pmatrix}}_{Q}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0$$

Thm: every quad. surf. $\subset \mathbb{P}^3$ is *projectively equiv.* to either

irreducible
① $x^2+y^2+z^2+w^2 = 0$
② $x^2+y^2+z^2 = 0$

reducible
③ $x^2+y^2 = 0$
④ $x^2 = 0$

Cor: quad surf smooth $\Leftrightarrow$ $\det(Q) \neq 0$.

Dfn: $C \subset \mathbb{P}^2$ curve, $f(x,y,z) = 0$. A *cone* on C is a surface in $\mathbb{P}^3$ defined by $f(x,y,z) = 0$.
↳ has singular point $[0:0:0:1]$

8.3 Segre Embedding

Dfn: *Segre Embedding* is the map $\phi: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$: $N = (n+1)(m+1) - 1$

$([x_0:\cdots:x_n],[y_0:\cdots:y_m]) \mapsto [\{x_i y_j\}_{i=0,\ldots,n,\, j=0,\ldots,m}]$

Prop: Segre is injective. Image of ϕ is variety given by vanishing of 2x2 determinants of matrix $[x_i y_j]$

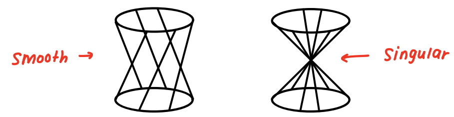
Prop: product of proj. var is proj. var.

Prop: smooth quad surf $\cong \mathbb{P}^1 \times \mathbb{P}^1$ as proj. var.

8.4 Lines on Surfaces

$S \subset \mathbb{P}^3$ irred. quad. surf.

- S smooth $\Rightarrow \cong \mathbb{P}^1 \times \mathbb{P}^1$, two rulings by lines
- S singular $\Rightarrow S$ = cone of smooth conic. one ruling by lines.



Thm (Segre) If $d \geq 3$, then S contains at most $(d-2)(11d-6)$ (finitely many) lines.
e.g. $d = 4$ contains ≤ 64 lines.

Thm (Cayley, Salmon): a smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

Prop: $S \subset \mathbb{P}^3$ irreducible cubic surface, $P \in S$ singular. Then $\exists\ \ell \subset S$ s.t $p \in \ell$.

Thm: $S \subset \mathbb{P}^3$ irreducible singular surface. Then either

- S has infinitely many singular points
- $S \cong$ cone on irreducible cubic (one singular point)
- S contains fewer than 27 lines.

9.2 : 27 Lines on a Smooth cubic surface

Rem: a plane and a line always intersect in $\mathbb{P}^3$.

Lem: $S \subset \mathbb{P}^3$ irreducible cubic surface, and Π a plane.
Then $\Pi \cap S$ is a cubic (curve) $C \subset \Pi$ and so is either:

① C an irreducible cubic
② C union of conic and line
③ C union of 3 lines

reducible

↳ S smooth + ③ ⇒ 3 distinct lines

Lem: $\ell \subset S$ a line. Then $\exists$ plane $\Pi \subset \mathbb{P}^3$, $\ell \subset \Pi$ s.t $\Pi \cap S$ is a union of 3 lines.

Lem: $\ell \subset S$. An ℓ'cs intersecting ℓ is contained in a plane w/ ℓ.

9.3 Rational Surfaces

$X :=$ proj. var, $\dim(X) = n$.

Dfn: X is *rational* if $\exists$ birat. map $\mathbb{P}^n \dashrightarrow X$

slightly weaker notion.

Dfn: X is *unirational* if $\exists$ dom rat. map $\mathbb{P}^n \dashrightarrow X$

Rational ⇒ unirational.

morphism of projective var.

Prop: $C \subset \mathbb{P}^2$ curve. A rational map $f: \mathbb{P}^1 \dashrightarrow C$ is regular

Thm (Lüroth): unirational curve is rational.

EXAMPLES

- irreducible conic is rational
- irreducible singular cubic is rational
- smooth cubic is NOT rational

Thm (Castelnuovo): unirational surface is rational

Thm: smooth cubic surface is rational.

10.1 : Tangent Spaces to Varieties

$X: f(x_1,\dots,x_n)=0$:= irreducible hypersurface in $\mathbb{C}^n$

Dfn: $P\in X$ is singular if $\frac{\partial f}{\partial x_1}(P)=\dots=\frac{\partial f}{\partial x_n}(P)=0$ otherwise smooth. $P=(a_1,\dots,a_n)\in X$ smooth, tangent plane is the affine hyperplane $\frac{\partial f}{\partial x_1}(P)(x_1-a_1)+\dots+\frac{\partial f}{\partial x_n}(P)(x_n-a_n)=0$

Similarly for in $\mathbb{P}^n$.

Prop : $X: f(x_1,\dots,x_n)=0$ irred. hypersurface in $\mathbb{C}^n$. The set of singular points is a proper alg. sub of X. The set of smooth points is dense.

$X\subset\mathbb{C}^n$ aff. var, $I(X)=\langle f_1,\dots,f_m\rangle$, $P=[a_1:\dots:a_n]\in X$.

Dfn : tangent space T_PX = affine subspace of $\mathbb{C}^n$ given by

$$\left.\frac{\partial f_i}{\partial x_1}(P)(x_1-a_1)+\dots+\frac{\partial f_i}{\partial x_n}(P)(x_n-a_n)=0\ \right\}\ i=1,\dots,m$$

Dfn: $P\in X$ smooth if $\dim T_PX=\dim X$.

If $f:X\to Y$ an iso of aff var, P smooth $\Leftrightarrow$ $f(P)$ smooth

Prop: X = proj. var. Set of singular points of X is a proper algebraic subset.

10.2 Blowups and Curves

Dfn: Parametrize $\mathbb{C}^2\times\mathbb{P}^1$ by $((x,y),[\alpha:\beta])$. The blowup of $\mathbb{C}^2$ at $(0,0)$ is the subset $Bl_{(0,0)}\mathbb{C}^2\subset\mathbb{C}^2\times\mathbb{P}^1$ defined by

$$x\beta=\alpha y$$

Let $\pi: Bl_{(0,0)}\to\mathbb{C}^2$ (proj. onto first factor), and denote by $E=\pi^{-1}(0,0)$ the exceptional curve.

Rem: $E\cong\mathbb{P}^1$, π is a morphism, π restricts to an isomorphism $S/E\to\mathbb{C}^2/(0,0)$, in particular, π is birational.

Dfn : Parametrize $\mathbb{P}^2\times\mathbb{P}^1$ by $([x:y:z],[\alpha:\beta])$. The blowup of $\mathbb{P}^2$ at $[0:0:1]$ is the subset $Bl_{[0:0:1]}\subset\mathbb{P}^2\times\mathbb{P}^1$ defined by

$$x\beta=\alpha y$$

Let $\pi: Bl_{[0:0:1]}\to\mathbb{P}^2$ (projection) and $E=\pi^{-1}([0:0:1])$ be called the exceptional curve.

Rem: $E\cong\mathbb{P}^1$, $\pi: Bl_{[0:0:1]}\to\mathbb{P}^2$ birational morphism. and $Bl_{[0:0:1]}\mathbb{P}^2\setminus E\xrightarrow{\sim}\mathbb{P}^2\setminus[0:0:1]$

Dfn: $C\subset\mathbb{P}^2$ a curve. Its strict transform $\tilde{C}\subset Bl_{[0:0:1]}\mathbb{P}^2$ is the closure of $\pi^{-1}(C\setminus[0:0:1])$. If $[0:0:1]\in C$, then $\tilde{C}=Bl_{[0:0:1]}C$ is the blowup of C at $[0:0:1]$.

Rem: blowup at smooth point is an iso, $C\cong\tilde{C}$.

idea: makes singular curves (eventually) smooth.

10.3 : Blowups and Surfaces

let $Z\subset\mathbb{C}^n$ be subvar, $I(Z)=(g_0,\dots,g_k)$

Dfn: blowup of $\mathbb{C}^n$ with center at Z is the subvariety $Bl_Z\mathbb{C}^n\subset\mathbb{C}^n\times\mathbb{P}^k$ defined by $u_ig_j(x)-u_jg_i(x)=0$ for $i\neq j$ and $[u_0:\dots:u_k]\in\mathbb{P}^k$.

$\pi: Bl_Z\mathbb{C}^n\to\mathbb{C}^n$: $\pi^{-1}(Z)\subset Bl_Z\mathbb{C}^n$ is the exceptional hypersurface

Thm: The blowup of $\mathbb{P}^2$ at 6 points, no 3 collinear and not all lying in a conic, is a cubic surface. Any cubic surface is obtained in this way.

10.4 : Birational Geometry

Thm: smooth projective curve has genus $g\geq 0$, and for each genus there are finitely many parameters describing a curve of genus g.

Rem: $f: C_1\dashrightarrow C_2$ rational map of smooth proj. curves. Then f is a morphism, and smooth birational curves are isomorphic.

Thm: $f: X\dashrightarrow Y$ rational morphism. Then there is a blowup $\pi:\hat{X}\to X$ and a morphism $\hat{f}:\hat{X}\to Y$ s.t $\hat{f}=f\circ\pi$.

$$\begin{tikzcd} & \hat{X} \arrow[dl, "\pi"'] \arrow[dr, "\hat{f}"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$